Structure of Max-Plus Hemispaces

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1 Introduction

We recall that $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$, with operations $\oplus = \max$, $\otimes = +$, and that $\mathbb{R}^n_{\max} = \mathbb{R}_{\max} \times ... \times \mathbb{R}_{\max}$ (n times) with the operations $X \oplus Y = (x_1 \oplus y_1, ..., x_n \oplus y_n)$ and $\alpha \otimes X = \alpha X = (\alpha x_1, ..., \alpha x_n)$. A line segment in max-plus between $X, Y \in \mathbb{R}^n_{\max}$ would be,

$$[X, Y] = \{\alpha X \oplus \beta Y | \alpha \oplus \beta = e\} = \{\max(\alpha + X, \beta + Y)| \ \max(\alpha, \beta) = 0\}. \tag{1}$$

We recall from [1] that a line segment in $\mathbb{R}^2_{\max}$, and consiquently $\mathbb{R}^n_{\max}$ has a similar case, has one of the three following forms:
When $X \leq Y$ and $x_1 - y_1 \leq x_2 - y_2$, then

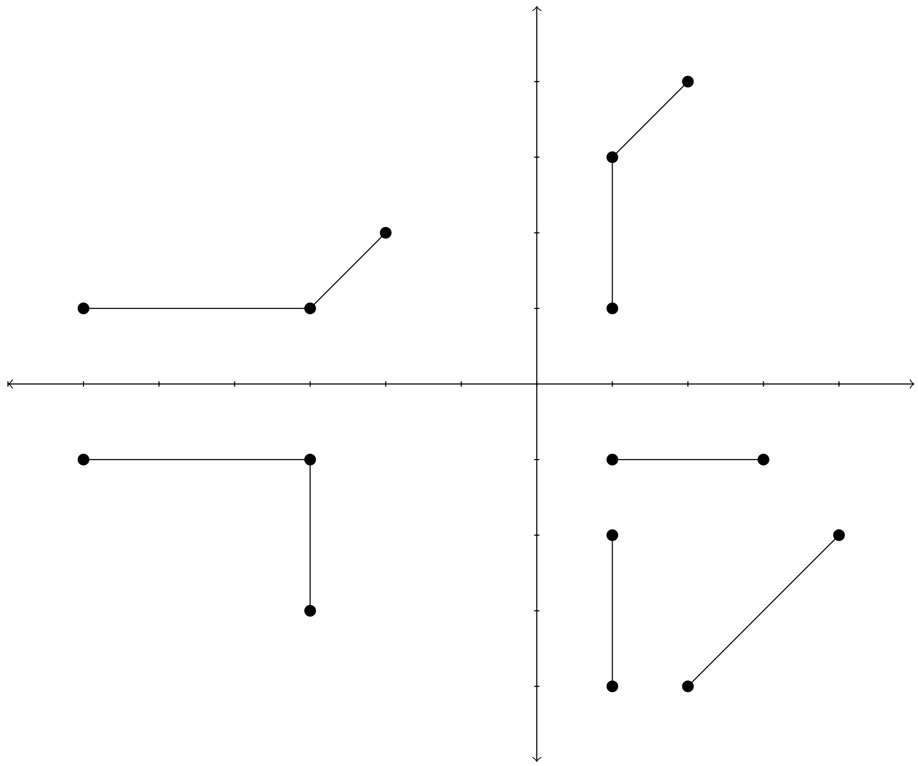
$$[X, Y] = [X, (y_1 + x_2 - y_2, x_2)] \cup [(y_1 + x_2 - y_2, x_2), Y] \tag{2}$$

When $X \leq Y$ and $x_2 - y_2 \leq x_1 - y_1$, then

$$[X, Y] = [X, (x_1, y_2 + x_1 - y_1)] \cup [(x_1, y_2 + x_1 - y_1), Y] \tag{3}$$

When $X \nleq Y$ and $Y \nleq X$, then

$$[X, Y] = [X, \max(X, Y)] \cup [\max(X, Y), Y] \tag{4}$$



We notice that are first possible line segment is given in the first quadrant, the second possible line segment is in the second quadrant, and the third line segment is in the third segment. The fourth quadrant contains the line segments where $x_1 = y_1$, $x_2 = y_2$, and $y_1 - x_1 = y_2 - x_2$.

A subset $S \subseteq \mathbb{R}^n_{\max}$ is said to be convex if $[x, y] \subseteq S$ for all $x, y \in S$. We recall from [1], that $S \subset \mathbb{R}^n_{\max}$ is a semispace at $z \in \mathbb{R}^n_{\max}$ if S is a maximal convex subset of $\mathbb{R}^n_{\max}$ avoiding z. For the set S to be a maximal convex set, then there exist no convex set $S' \neq S$, where $S \subset S'$ and $z \notin S'$. Note that for finite points we can take z to be the point $(0, 0, ..., 0)$, since a semispace at any other point results from a translation of the semispaces around the origin. Nitica and Singer [1] determined that there are exactly $n+1$ semispaces at z, namely

$$S_0 = \{X \in \mathbb{R}^n_{\max} | 0 < \max(x_1, ..., x_n)\} = \{X \in \mathbb{R}^n_{\max} | 0 < x_1\} \cup ... \cup \{X \in \mathbb{R}^n_{\max} | 0 < x_n\} = \{X \in \mathbb{R}^n_{\max} | X \not\leq 0\} \tag{5}$$

$$S_k = \{X \in \mathbb{R}^n_{\max} | x_k < \max(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n, 0)\}$$

$$= \{X \in \mathbb{R}^n_{\max} | x_k < 0\} \cup ... \cup \{X \in \mathbb{R}^n_{\max} | x_k < x_{k-1}\} \cup \{X \in \mathbb{R}^n_{\max} | x_k < x_{k+1}\} \cup ... \cup \{X \in \mathbb{R}^n_{\max} | x_k < x_n\}. \tag{6}$$

In this paper we determine the hemispaces in $\mathbb{R}^n_{\max}$. To do so, we considered the complements of the semispaces centered at the origin. By [1] these complements are given by

$$\complement S_0(z) = \{X \in \mathbb{R}^n_{\max} | 0 \geq \max(x_1, ..., x_n)\} \tag{7}$$

$$\complement S_k(z) = \{X \in \mathbb{R}^n_{\max} | x_k \geq \max(x_1, ..., x_{k-1}, x_{k+1}, ..., x_n, 0)\}. \tag{8}$$

Since these complements intersects one another at their boundary points, we consider them without boundaries, and then determined how we could partition $\mathbb{R}^n_{\max}$ into two convex subsets made up of semispace complements and their boundaries.

Definition 1.1 A set $H \subseteq \mathbb{R}^n_{\max}$ is a hemispace in $\mathbb{R}^n_{\max}$ if both H and $\complement H$ are convex.

Let S_i be a semispace and $\complement S_i$ be the complement of that semispace where x_i is the greatest variable in the space and $x_0 = 0$. For notational ease, we refer to $\complement S_i / bd\complement S_i$ in the rest of the paper by $\complement S_i$.

Definition 1.2 A face of dimension k will be refered to as a $k - face$ for the remainder of the paper.

Definition 1.3 A hyperplane of the form:

$$x_0 \oplus ... \oplus x_n = x_m \oplus ... \oplus x_n; m \leq n \tag{9}$$

has a boundary hyperplane of:

$$x_0 \oplus ... \oplus x_{m-1} = x_m \oplus ... \oplus x_n$$

And has the section where $x_m, ..., x_n > x_0, ..., x_{m-1}$ filled.

2 Hemispaces in $\mathbb{R}^2_{\max}$

We note that $\complement S_i$ is convex with or without the boundary. We also know that semispaces are the building blocks of hemispaces.

Lemma 2.1 *$\complement S_i / bd\complement S_i$ is a minimal non-trivial hemispace.*

Proof Let H be a minimal non-trivial hemispace. We first consider H to be a single point, $H = \{P\}; p = (p_1, ..., p_n)$. However, P is not a hemispace since $\mathbb{R}^n_{\max}/\{P\}$ is not convex because there exist points $Q = (q_1, p_2, ..., p_n)$ and $R = (r_1, p_2, ..., p_n)$ in $\mathbb{R}^n_{\max}/\{P\}$ such that $q_1 < p_1 < r_1$, so $P \in [Q, R]$. Thus, we consider adding a point Q to H. To maintain convexity in H, we must have $[P, Q] \subseteq H$ as well. If $[P, Q]$ is not degenerate (has 2 conecting segments for the case $n = 2$, then clearly both of those segments must be made half-lines to maintain the convexity of $\complement H$. Otherwise there exist $S, T \in \complement H$ located on the full line containing one of those segments, clearly a non-empty $A \subseteq [P, Q]$ also has $A \subseteq [S, T]$. Now we have two half lines that start at,

$$(q_1 + p_2 - q_2, p_2) \text{ if } Q < P$$

$$(p_1, q_2 + p_1 - q_1) \text{ if } P < Q$$

$$\max(P, Q) \text{ if } P \nleq Q,\ Q \nleq P$$

and extend through P and Q. Note that this creates the boundary of a semispace. Clearly, to maintain convexity, H must also contain the entire space either above or below these half lines. One side gives a semispace, and the other gives its complement. Clearly the complement is smaller, which we can make even smaller by excluding the boundary.

If the segment connecting P and Q is degenerate, then for the case $n = 2$, either $q_1 = p_1$ or $q_2 = p_2$. Without loss of generality, suppose $q_2 = p_2$ and $q_1 > p_1$, so the line line connecting them is a horizontal line. Clearly we must extend this segment to a half-line and include the space either above or below it. If we extend it to the left and include the space below it, we get $\complement S_0$. If we extend left and include the space above the line, then it is clear we must also include half-line starting at Q parallel to the main bisector to maintain the convexity of $\complement H$, and we end up with $\complement S_2$. Finally if we extend to the right, and include below, we must also have the half line starting at P, parallel to the main bisector, and we get $\complement S_1$. If we include above, it is clear that $\complement H$ is not convex. The only way to fix this problem is to include the other vertical (or horizontal) half-line starting at P and the section to the right (or above) it. This is the only hemispace that can be formed without being the union of semispace complements, but it is clear that these hemispace are larger then any single semispace complement.

Remark 2.2 Because of this previous lemma, it makes sence to look at the semispaces, and hence there complements, when trying to determain how a hemispace would work. Let us look at $\mathbb{R}^2_{\max}$ to get an understanding of how to form these hemispaces. We notice that in $\mathbb{R}^2_{\max}$ we have three semispaces and their complements. Then we are looking at

$$\complement S_0 = \{X \in \mathbb{R}^2_{\max} | 0 \geq \max(x_1, x_2)\} \tag{10}$$

$$\complement S_1 = \{X \in \mathbb{R}^2_{\max} | x_1 \geq \max(0, x_2)\} \tag{11}$$

$$\complement S_2 = \{X \in \mathbb{R}^2_{\max} | x_2 \geq \max(0, x_1)\} \tag{12}$$

and their boundaries to form these hemispaces. We notice that $\complement S_0$ is the negative quadrant with the boundry of $\complement S_0$ being the $1 - faces$ $(x_1, 0); x_1 < 0$ and $(0, x_2); x_2 < 0$ along with the origin.

Example 2.3 The first basic hemispaces in $\mathbb{R}^2_{\max}$ that we come across is S_0 and $\complement S_0$. What we want to see is how we can split and distribute the boundary between them such that both S_0 and $\complement S_0$ remain convex and by that, hemispaces.

Let us define $A = S_0$ and $B = \complement S_0$ without the boundary. $l_1 = (x_1, 0); x_1 < 0$, $l_2 = (0, x_2); x_2 < 0$, and $0 = (0, 0)$. We immediately have two basic cases were $l_1, l_2, 0 \in A$ or $l_1, l_2, 0 \in B$.
Now we can see that if both $l_1, l_2 \in A$ then for A to stay convex $[l_1, l_2] \in A$
$[l_1, l_2] = [(x_1, 0), (0, x_2)] = \{(\max(\alpha + x_1, \beta), \max(\alpha, \beta + x_2))\}$
since $x_1, x_2, \alpha, \beta \leq 0$ then $\{(\max(\alpha + x_1, \beta), \max(\alpha, \beta + x_2))\} = \{(0, 0)\}$ which implies that $0 \in [l_1, l_2] \in A$.
This is symmetric to $[l_1, l_2] \in B$.

Lemma 2.4 *If $l_1, l_2 \in \mathbb{R}^2_{max}$ are in the same convex space, then 0 is in that convex space.*

Proof The proof of Lemma 2.4 follows directly from our previous example.

Let us now consider the case were l_1 and l_2 are not in the same convex space. Let $l_1 \in A$. Then consider the line segment $[(-2, 0), (2, 0)]$ with $(-2, 0) \in l_1$ and $(2, 0) \in A$.
$[(-2, 0), (2, 0)] = \{(\max(\alpha - 2, \beta + 2), \max(\alpha, \beta))\}$
if we let $\alpha = 0$ and $\beta = -2$, then $\{(\max(\alpha - 2, \beta + 2), \max(\alpha, \beta))\} = \{(0, 0)\}$. Which implies that $0 \in [(-2, 0), (2, 0)] \in A$. Similarly, if $l_2 \in A$ then $0 \in A$.

Remark 2.5 Notice that this result say's that if either l_1 or $l_2 \in A$ then $0 \in A$. So flipping these results we notice that if $0 \in B$, then neither l_1 or l_2 can be in A, therefore they both must be in B. Giving us a trivial result with the entire boundary in B.

Because of this result and Lemma 2.4 all possible splits of the boundary in $\mathbb{R}^2_{max}$ for S_0 are:
$l_1, l_2, 0 \in A$
$l_1, l_2, 0 \in B$

$l_1, 0 \in A$ and $l_2 \in B$
$l_2, 0 \in A$ and $l_1 \in B$

We have a similar case when considering $S_1, \complement S_1$ and $S_2, \complement S_2$.

Remark 2.6 It is easy to see that some of these properties for $R^2_{\max}$ could be generalized to work for $R^n_{\max}$. But as you may notice for the case were $n = 2$, since we only have three semispace complements, we are always only seperating one of those complements from the rest. In section 3 we continue to work on the one section split in $R^n_{\max}$ and in section 4 we look at what will happen if we split more then one section apart, which can not happen unless $n \geq 3$.

3 Spliting the Boundary of one Sector in $\mathbb{R}^n_{\max}$

Recall from [2] that the boundary of $\complement S_i$ is a hyperplane. In this section we consider how to split one of these hyperplanes between S_i and $\complement S_i$ such that we have two hemispaces. Without loss of generality let us look at S_0 and $\complement S_0$. We know from the last section that for $n = 2$ if we have the origin in $\complement S_0$ then all the conecting one dimentional lines are in $\complement S_0$. We can actually generalize this result to $R^n_{\max}$. First let us make neccesary definitions.

Definition 3.1 $X = (x_1, x_2, .., x_n)$ is a $m - face$ with $m < n$ if there exist exactly m unique coordinates in X, all non-unique coordinates either are equal to zero or the same x_i where x_i is one of the unique coordinates.

Remark 3.2 If the non-unique coordinates in X are all dependent on a unique x_i then this $m - face$ is connected to the one dimensional main bisector. If the non-unique coordnates in X are equal to 0, then the $m - face$ is connected to the negitive octant.

Definition 3.3 An $m + 1 - face$ Y is said to be adjacent to X iff the same m unique coordinates in X are also unique in Y, and Y has one additional unique coordinate.

Example 3.4 Let $X = (x_1, x_2, x_3, 0, 0, 0)$ and let $Y = (y_1, y_2, y_3, 0, y_5, 0)$. X, a $3 - face$, and Y, a $4 - face$, are adjacent because in both X and Y, x_i and y_i are independant coordnates for $i \in [1, 2, 3]$, while x_5 is 0 in X and y_5 is an independant coordnate in Y.

Remark 3.5 An $m - face$ X can not be adjacent to a $m + k - face$ Y, where $|k| \geq 2$.

Lemma 3.6 *If a $m - face$ with $m < n$ is a part of the boundary of $\complement S_i$, then all adjacent $m + 1 - faces$ are in $\complement S_i$.*

Proof Let X be an $m-face$ with $m<n$, and $X \in \complement S_i$. Let Y be a $m+1-face$ and adjacent to X.
Assume $Y \in S_i$
We want to show that $\exists Z \in S_i$ s.t. $X \in [Y,Z]$
Let $X=(x_{1_a}, x_{2_a}, ..., x_{m_a}, 0, ..., 0)$. Then, $Y=(x_{1_b}, x_{2_b}, ..., x_{m_b}, y_1, y_2, ..., y_k)$, where $m+k=n$, and $\exists y_i \in \mathbb{R}_{\max}, i \in [1,2,...,k], y_i \neq 0$, and $y_j=0, j \neq i$ and $j \in [1,2,...,k]$.
Let $Z=(x_{1_b}, x_{2_b}, ..., x_{m_b}, -y_{1_a}, -y_{2_a}, ..., -y_{k_a})$. Where $-y_{i_a}$ is chosen s.t. $-y_{i_a}$ is the same sign as $-y_i$, and $Z \in S_i$.
$[Y,Z]=[(x_{1_b}, x_{2_b}, ..., x_{m_b}, y_1, y_2, ..., y_k), (x_{1_b}, x_{2_b}, ..., x_{m_b}, -y_{1_a}, ..., -y_{k_a})]=$
$\{(\max(\alpha+x_{1_b}, \beta+x_{1_b}), ..., \max(\alpha+x_{m_b}, \beta+x_{m_b}), \max(\alpha+y_1, \beta-y_{1_a}), ..., \max(\alpha+y_k, \beta-y_{k_a}))\}$
$\max(\alpha+x_{i_b}, \beta+x_{i_b})=x_{i_b}; i \in [1,2,...,m]$
When $y_j=0$, then $\max(\alpha+y_j, \beta-y_{j_a})=0; j \in [1,...,k]$
When $y_j \neq 0$, then $\max(\alpha+y_j, \beta-y_{j_a})=0$, when $\alpha=0$ and $\beta=y_{j_a}$ if $y_{j_a}<0$ or when $\alpha=-y_j$ and $\beta=0$ is $y_j>0$.
Then $X \in [Y,Z]$.

This result immediatly gives a corollary about a one sector split in $\mathbb{R}^n_{\max}$.

Corollary 3.7 *If the center is part of the boundary containing only $\complement S_i$, then we have the trivial hemispace were the entire boundary is containing only $\complement S_i$*

Remark 3.8 Corollary 3.7 tells us that all non-trivial hemispaces for a one sector split of the boundary is going have to have the origin part of the boundary of S_i. We can recal from Lemma 2.2, that no more the one line of dimension one can be in the same convex section without the origin for $n=2$. This brings us to are next lemma and a generalization of this result for $\mathbb{R}^n_{\max}$.

Lemma 3.9 *If the center and all faces up to dimension $m<n$ are in S_i, then no more then one $m+1-face$ is in $\complement S_i$.*

Proof Let the center and all faces up to dimension $m<n$ be in S_i. Let $T=$ the number of $m+1-faces$ that are in $\complement S_i$. Assume $T \geq 2$. Since $T \geq 2$ then we can choose two faces of degree $m+1$, E_1 and $E_2 \in \complement S_i$.
Let $E_1=\{(x_1, x_2, ..., x_n) | x_i \neq 0$ for exactly $m+1$ $i's$, and $x_j=x_i$ or $x_j=0$ for the remaining $n-(m+1)$ points$\}$ and $E_2=\{(y_1, y_2, ..., y_n) | y_i \neq 0$ for exactly $m+1$ $i's$, and $y_j=y_i$ or $y_j=0$ for the remaining $n-(m+1)$ points$\}$ and $E_1 \neq E_2$.
$[E_1, E_2]=[(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)]=\{(\max(\alpha+x_1, \beta+y_1), ..., \max(\alpha+x_n, \beta+y_n))\}$
Case 1: $E_1<0<E_2$ similarly $E_2<0<E_1$
Case 2: $0<E_1<E_2$ similarly $0<E_2<E_1$
Case 3: $E_1 \nleq E_2$ and $E_2 \nleq E_1$

Case 1: Let $\alpha=0$ and $\beta=-\max(y_i | i \in [1,2,...,n])$ then $[E_1, E_2]=\{(\max(x_1, y_1-y_i), ..., \max(x_i, y_i-y_i), ..., \max(x_n, y_n-y_i))\}$. This has at most m unique points since the i^{th} point was changed to 0. Then $\exists k \in K$ a $p-face$ with $p<m+1$ s.t. $k \in [E_1, E_2]$. Therefore, both E_1 and E_2 cannot be in $\complement S_i$.

Case 2: Let $0 \leq x_n \leq ... \leq x_1 \leq y_1 \leq ... \leq y_n$ and let $\alpha=0$.
$[E_1, E_2]=\{(\max(x_1, \beta+y_1), ..., \max(x_n, \beta+y_n))\}$. Since $y_n>x_1>0$, then $\exists \beta_0$ s.t. $y_n+\beta_0=x_1$ and $\max(x_i, \beta+y_i)=x_i; y_i<y_n$. Then, $[E_1, E_2]=\{(x_1, x_2, ..., x_i, x_1, x_1, ..., x_1)\}$. Since we made some independent variables dependent on x_1, then $\exists k \in K$ a $p-face$ with $p<m+1$ s.t. $k \in [E_1, E_2]$. Therefore, both

E_1 and E_2 cannot be in $\complement S_i$.

Case 3: Since $E_1 \nleq E_2$ and $E_2 \nleq E_1$, then $[E_1, E_2] = [E_1, \max(E_1, E_2)] \cup [\max(E_1, E_2), E_2]$. Let $x_i > y_i; i \in [1, ..., r]$ and $y_j > x_j; j \in [r+1, ..., n]$.
Then, $[E_1, \max(E_1, E_2)] = [(x_1, ..., x_n), (x_1, ..., x_r, y_{r+1}, ..., y_n)] = \{(x_1, ..., x_r, \max(\alpha+x_{r+1}, \beta+y_{r+1}), ..., \max(\alpha+x_n, \beta+y_n))\}$.
Let $r < m+1$, if $r \geq m+1$ look at the line segment $[\max(E_1, E_2), E_2]$. Rest of the proof is similar to case 2.

Therefore $T < 2$.

Remark 3.10 With Lemma 3.6 and Lemma 3.9 we can determain all boundary splits between S_i and $\complement S_i$, both trivial and non-trivial, for $\mathbb{R}^n_{\max}$ by first choosing where our origin lies, and then using the lemma's to determain where higher dimensional faces must fall. Once we stubble upon a face that can go on either face, we choose were we want this face to fall, and once again the Lemma's show that more higher dimensional faces will follow. We continue this process until all faces are acconted for. We go over all of the combinatorics of this later in this paper.

4 Multi-Sector Split in $R^n_{\max}$

Since $\mathbb{R}^n_{\max}$ has $n+1$ sectors as mentioned earlier. The first n that can have a multi-sector split would be $n = 3$. For looking at how these splits can happen we use two lemma's, without proof, for a general idea. We will later show and algebraic result with proof that works for all types of splits in all dimensions.

Lemma 4.1 *Let a face have k sectors. Then if all $k-1-faces$ are bordering that section, then the whole boundary borders that section.*

Lemma 4.2 *If all faces up to dimension $m < n$ are bordering the same section, then no more then one $m+1-face$ can border the other section.*

Remark 4.3 Lemma 4.1 and Lemma 4.2 give general rules to follow when spliting the boundary of a k sector split but this leaves a lot of choices undetermained. For these reason we took a different approch to solving this problem using some set theory and combinatorics. We begin with a lemma showing that the boundary must stay connected and a line segment on the boundary stay's on the boundry, while also introducing some new notation, which will lead us to our main theorem.

Lemma 4.4 *Let $H \subseteq R^n_{\max}$ be a hemispace and $S \subseteq R^n_{\max}$ be a $k-face$. If there exists $p \in S$ such that $p \in H$, then $S \subseteq H$.*

Proof Observe that it suffices to show that for a given $q \in S$, either for each $\complement S_i$ there exists $r_i \in \complement S_i$ such that $p \in [q, r_i]$ or for each $\complement S_i$ there exists $s_i \in \complement S_i$ such that $q \in [p, s_i]$ since this would establish that p and q must be in the same hemispace. Furthermore, we must only find such r_i or s_i for each $\complement S_i$ that is bounded by S because if all such $\complement S_i$ are in H (or all not in H), then S is not a boundary of H. We will do this

by showing that each point in the lines that go through p and are parallel to $1-faces$ (the axes or main bisector) that bound S must be in S. This process will be independent of p, and so once we know that a half-line going through p and parallel to one $1-face$ is in H, we can then repeat the the process for a line parallel to a different $1-face$ for each point on that half line, thus tracing out a $2-face$ cross section of the $k-face$, and so on until we trace out the entire $k-face$ with half-lines parallel to the k $1-faces$ that S is bounded by.
First, suppose S is of the form $x_{k+1}=x_{k+2}=...=x_n>0, x_1,...,x_k$, so $p=(p_1,...,p_k,p_0,...,p_0)$ where $p_0>p_1,...,p_k$. To show that every point in S on the half-line that goes through p and is parallel to the x_i axis must also be in H, consider an given point $q\neq p$ on that half-line, so $q=(p_1,...,p_{i-1},q_i,p_{i+1},...,p_k,p_0,...,p_0)$ where $p_0>p_1,...,p_{i-1},q_i,p_{i+1},...,p_k$. Observe that since p and q only differ by one coordinate, we must have either $p>q$ or $p<q$. Without loss of generality, assume $q>p$ (i.e. $q_i>p_i$); the other case follows from interchanging p and q. For each $j\in\{k+1,...,n\}$ choose $r_j=(p_1,...,p_{i-1},q_i,p_{i+1},...,p_k,0,...,0,p_0,0,...,0)$ where the p_0 is the jth coordinate so that $r_j\in\complement S_j$. Thus, $\max(r_j,p)=q$, so $q\in[r_j,p]$.
Now, consider a given q on the half-line through p that is parallel to the main bisector, so $q=(p_1+c,...,p_k+c,p_0+c,...,p_0+c)$. Without loss of generality, assume $q>p$, i.e. $c>0$. Then for each $j\in\{k+1,...,n\}$ choose $r_j=(p_1,...,p_k,0,...,0,p_0,0,...,0)$ where the p_0 is the jth coordinate so that $r_j\in\complement S_j$. Then $\max(\alpha r_j,\beta q)=p$ for $\alpha=0$ and $\beta=-c$, so $p\in[r_j,q]$.
Now, suppose S is of the form $0=x_{k+1}=...=x_n>x_1,...,x_k$, so $p=(p_1,...,p_k,0,...,0)$ where $0>p_1,...,p_k$. Let $q\neq p$ be a given point on the half-line passing through p and parallel to the x_i axis, so $q=(p_1,...,p_{i-1},q_i,p_{i+1},...,p_k,0,...,0)$. Without loss of generality, assume $q>p$, i.e. $q_i>p_i$. For each $j\in\{k+1,...,n\}$, choose $r_j=(p_1,...,p_{i-1},q_i+1,p_{i+1},...,p_k,0,...,0,1,0,...,0)$ where the 1 is the jth coordinate so that $r_j\in\complement S_j$. Thus, $\max(\alpha r_j,\beta p)=q$ for $\alpha=-1$ and $\beta=0$, so $q\in[r_j,p]$. Finally, for $\complement S_0$, choose $r_0=(p_1,...,p_{i-1},q_i,p_{i+1},...,p_k,-1,...,-1)\in\complement S_0$. Then $\max(r_0,p)=q$, so $q\in[r_0,p]$.

Remark 4.5 The basic principle that this proof is saying is that if any general point of a $k-face$ is part of a hemispace, then the entire $k-face$ is part of that hemispace. This also means that you can not split a $k-face$ between two hemispaces. This fact has been taken for granted up until now, but it is important to prove for the next theorem.

Lemma 4.6 *Suppose S_1,S_2 are faces in $\mathbb{R}^n_{\max}$ of dimension K and L respectively, bordering the same hemispace, with $k\leq l\leq n$. Suppose the highest dimensional boundary of S_1 and S_2 is S_3 with dimension $m\leq k$. Then for $p\in S_1$ and $q\in S_2$, the segment $[p,q]\subseteq S_1\cup S_2\cup S_3$ and $[p,q]\cap S_i\neq\emptyset; i\in\{1,2,3\}$.*

Proof For notational purposes, consider 0 to be another variable x_i, so that there are a total of $n+1$ variables. By permuting variables if necessary, observe that S_1 is of the form
$x_{k+1}=x_{k+2}=...=x_n=x_{n+1}>x_1,...,x_m,x_{m+1},...,x_k$ and S_2 is of the form
$x_{m+1}=...=x_k=x_{l+k-m+1}=x_{l+k-m+2}=...=x_n=x_{n+1}>x_1,...,x_m,x_{k+1},...,x_{l+k-m}$. Thus, S_1 has k free variables $\{x_1,...,x_k\}$ and S_2 has l free variables $\{x_1,...,x_m,x_{k+1},...,x_{l+k-m}\}$, and they share the free variables $\{x_1,...,x_m\}$. Hence $p\in S_1$ is of the form $p=(x_1,...,x_k,x_0,x_0,...,x_0)$ where there are k x_0s and $q\in S_2$ is of the form $q=(y_1,...,y_m,y_0,...,y_0,y_{k+1},...,y_{l+k-m},y_0,...,y_0)$ where there are $k-m$ y_0s between y_m and y_{k+1} and $n+1-(l+k-m)$ y_0s after y_{l+k-m}, where $x_0>x_1,...x_k$ and $y_0>y_1,...,y_m,y_{k+1},...,y_{l+k-m}$. Without loss of generality assume $x_0>y_0$, and let $d=y_0-x_0$. Let $\alpha=0$, then $\max(\alpha p,\beta q)\in S_1$ $\forall\beta\leq 0$. Let $\beta=0$, then $\max(\alpha p,\beta q)\in S_1$ $\forall\alpha>d$, $\max(\alpha p,\beta q)\in S_3$ for $\alpha=d$, and $\max(\alpha p,\beta q)\in S_2$ $\forall\alpha<d$. Thus, by definition of line segments, $[p,q]\subseteq S_1\cup S_2\cup S_3$ and $[p,q]\cap S_i\neq\emptyset; i\in\{1,2,3\}$.

Remark 4.7 As mentioned earlier, Lemma 4.6 shows that the line segment between any two boundary points goes through the face that is the union of those two points. Now consider the set $A=\{0,1,...,n\}$ and it's powerset $P(A)=\{\emptyset,\{0\},...,\{1,2,...,n\},A\}$, where $\{i\}$ implies that $x_i>0,x_1,...,x_n$ and $\{i,j,k\}$

implies that $x_i = x_j = x_k > 0, x_1, ..., x_n$, also $x_0 = 0$. This means that $\emptyset$ is the whole space $\mathbb{R}^n_{\max}$ and A is the center, in our case the origin. Now since we have no reason to consider the entire space we introduce the set $B = P(A)/\emptyset$.

Theorem 4.8 *$H \subseteq B$ is a hemispace iff H and B/H are closed under unions.*

Proof Let H be a hemispace, this implies that H in convex and that all line segments in H remain in H. This means that if we take two points in H, by Lemma 4.6 and the fact that H is a hemispace, the union of the two planes that those points were on is in H. Since H is a hemispace the B/H is a hemispace, then B/H is also closed under unions.
Let H and B/H be closed under unions. Then for any two points in H the line segment between them stay's in H by Lemma 4.6. Similar for B/H. This implies that H and B/H are convex. Since H and B/H are convex, then they are both hemispaces.

Remark 4.9 This theorem is not only true for a multi-section split but it can also be applied to a one section split. This gives all possibilites for all types of splits. In the next section we work on trying to count how many possible splits we have and how fast the number of splits grows as n grows.

5 Combinatorics

Corollary 5.1 *In $\mathbb{R}^n_{\max}$ there are exactly $2f(n)$ hemispaces at a finite point, where the function f is defined recursively by $f(0) = 1$ and*

$$f(n) = \binom{n+1}{1} f(n-1) + \binom{n+1}{2} f(n-2) + \cdots + \binom{n+1}{n-1} f(1) + \binom{n+1}{n} f(0) + 1. \tag{13}$$

Remark 5.2 The function f calculates the number of hemispaces that contain the origin. Thus, in order to calculate the total number of hemispaces we must multiply by 2 to count their complements as well.

Proof We will establish the corollary by induction on n. Clearly $f(1) = 3$ since the hemispaces in $\mathbb{R}^n_{\max}$ that contain the origin are the closed half-line to the right, closed half-line to the left, and the whole space. Now, suppose the formula holds for $n-1$ and we must show that it holds for n. Suppose H is a hemispace in $\mathbb{R}^n_{\max}$ containing the origin and G is its complement, and we will determine the total number of ways to construct H. Since the origin is in H, it follows that only one $1-face$ can be in G since having any two $1-faces$ in G would also require G to contain the origin to maintain convexity. Observe that there are $\binom{n+1}{1}$ $1-faces$ in $\mathbb{R}^n_{\max}$ that can be in G. Once a $1-face$ is chosen to be in G, then by the Theorem we know that every face containing the variable that is not in the $1-face$ must be in H since having any of these faces in G would also require the origin to be in G to maintain convexity. Thus, finding the total number of hemispaces containing the origin now simplifies to an equivalent problem in $\mathbb{R}^{n-1}_{\max}$, where G now contains the "origin" (the $1-face$). Hence, there are $\binom{n+1}{1}$ ways to reduce the problem to a $n-1$ dimensional problem, but the number of hemispaces in $\mathbb{R}^{n-1}_{\max}$ is given by $f(n-1)$, so this accounts for the $\binom{n+1}{1} f(n-1)$ term.
Alternatively, H could also contain every $1-face$. In this case, there can be no more than one 1 dimensional

in G since, by the Theorem, putting any two $2-faces$ in G would also require either the origin or a $1-face$ to be in G to maintain convexity. Thus, there are $\binom{n+1}{2}$ $2-faces$ that can be chosen to be in G, and doing so simplifies the problem to the case in $\mathbb{R}^{n-2}_{\max}$. This accounts for the $\binom{n+1}{2}f(n-2)$ term. Again, we could also choose for every $2-face$ to be in H. Following this formula, it is clear that there are $\binom{n+1}{d}$ ways to simplify the problem to the case of $\mathbb{R}^{n-d}_{\max}$, which accounts for all the terms but the last "1", which represents choosing to put every face in H, i.e. the trivial hemispace.

Remark 5.3 Using this equation to find the number of hemispaces we get a sequence of numbers starting at $f(0) = 1$. This sequence grows exponentially fast, it is also same sequences as the Ordered Bell Numbers excluding the first term of that sequence.

6 Hemispaces that are not Semispaces

We know from [2] that the closure of a hyperplane is a hemispace. It is also clear that a hyperplane that uses all variables availiable is a semispace or the boundary of a semispace. When each variable appears only once we have the boundary of a hemispace.

Example 6.1 A basic boundary in $\mathbb{R}^2_{\max}$ is $x_1 \oplus x_2 = 0$. This is the boundary of the negitive quadriant and a semispace.

In this section we will look at what happens when we don't use all of the variables, and what the boundaries look like in this case. In $R^2_{\max}$ we only have three hyperplanes that we can look at:

$$x_1 = 0$$

$$x_2 = 0$$

$$x_1 = x_2$$

These are very basic lines but they do form the boundary lines of hemispaces in $\mathbb{R}^2_{\max}$.

Remark 6.2 As with the rest of this paper we want to look at what the hemispaces of this form would look like and how we could split the boundary, if at all, to have them remain hemispaces. It is very clear from our example that if we have a hyperplane, like in the example, with only two variables. Then the boundary if only one $n-1-face$. Since we have proved earlier that one point in the face being part of the boundary means the whole face is in the boundary, then we can see that if there is only one face for the boundary, then we only have the to trivial cases of the entire boundary being with one or the other sections.

Definition 6.3 A hyperplane of the form:

$$x_i \oplus ... \oplus x_m = x_k \oplus ... \oplus x_m; 0 \le i < k \le m < n \tag{14}$$

is a hyperplane in $\mathbb{R}^n_{\max}$ with the boundary faces following the hyperplane:

$$x_i \oplus ... \oplus x_{k-1} = x_k \oplus ... \oplus x_m$$

and the section where $x_k, ..., x_m > x_i, ..., x_{k-1}$ is filled.

Lemma 6.4 *Let $x_0, x_1, ..., x_n$ be the variables in $\mathbb{R}^n_{\max}$, then any hemispaces (after permutating variables if necessary) in the form:*

$$x_i \oplus x_{i+1} \oplus ... \oplus x_{i+m} = x_j \oplus x_{j+1} \oplus ... \oplus x_{j+t} \tag{15}$$

where $i + m < j$ and $m + t < n - 1$, then the hemispaces with this hyperplane as a border acts exactly like hemispaces in $\mathbb{R}^{m+t+1}_{\max}$ with sections that are the union of semispace complements.

Remark 6.5 Since the space is going from $n \to m + t + 1$, then every dimensional space in the boundary of dimension k is treated as if it is a $k - ((n + 1) - (m + t + 2)) - face$. Now that we know how a hyperplane with $2 \leq k \leq n+1$ variables acts, it is important to know how many different hyperplanes there can be with k variables in dimension n ($f(k|n)$). The formula for that is:

$$f(k, n) = \binom{n+1}{2}\binom{n-k+2}{1} + \binom{n+1}{2}\binom{n-k+3}{2} + ... + \binom{n+1}{2}\binom{n-1}{k-2} \tag{16}$$

Where $f(2, n) = f(n + 1, n) = \binom{n+1}{2}$

7 Points at $-\infty$

After reordering the coefficents of the variables as needed, let $a_0 = a_1 = ... = a_l = 0 > -\infty = a_{l+1} = ... = a_n$. Then any hyperplane of the form:

$$a_0x_0 \oplus ... \oplus a_mx_m \oplus a_{l+1}x_{l+1} \oplus ... \oplus a_kx_k = a_{m+1}x_{m+1} \oplus ... \oplus a_lx_l \oplus a_{k+1}x_{k+1} \oplus ... \oplus a_nx_n \tag{17}$$

With $m < l < k < n$, can be reduced to:

$$x_0 \oplus ... \oplus x_m = x_{m+1} \oplus ... \oplus x_l \tag{18}$$

Since $x_i \oplus a_sx_s = x_i$ with $i \in [0, ..., l]$ and $s \in [l + 1, ..., n]$.

Remark 7.1 In this case we have similar results to what we had in section 6. The main difference is, instead of all of the $k - faces$ dropping down in dimension, we now are only focusing on the l dimensional cross section where $x_{l+1} = ... = x_n = -\infty$. This cross section follows the same rules with the splitting of the boundary as the rest of the paper did however, all of the higher dimensional faces that are not split have the ability to belong to either hemispace with no restrictions. Since these higher dimensional faces are not split and can belong to either hemispace, because that would still keep them convex by our earlier definition, we multiply our total number of hemispaces "$2f(n)$" by two. So for this case we would have $4f(l)$ hemispaces, including complements.

Any hyperplane boundary of the form:

$$a_1x_1 \oplus ... \oplus a_lx_l \oplus a_{k+1}x_{k+1} \oplus ... \oplus a_nx_n = a_{l+1}x_{l+1} \oplus ... \oplus a_kx_k \tag{19}$$

With $l < k \leq n$, can reduce to:

$$x_1 = x_2 = ... = x_l = ... = x_n = -\infty \tag{20}$$

Since we have that $x_1 \oplus ... \oplus x_l = \max(x_1, ..., x_l) = -\infty$.

Remark 7.2 The hemispace in this situation would be $x_1 \oplus ... \oplus x_l > -\infty$ in the l dimensional cross section with the point $(-\infty, ..., -\infty)$ as the boundary point. Leaving only the trivial hemispace of the whole space without the point and the whole space with the point. Once again the space outside of the cross section does not split and can belong to either hemispace with effecting convexity.

The final situation is when all the coefficents are equal to $-\infty$.
Then all of the variables are equal to $-\infty$.

Recall from [1] that this situation will give us the semispace in $\mathbb{R}^n_{\max}$:

$$S(-\infty) = \mathbb{R}^n_{\max}/\{(-\infty, ..., -\infty)\}. \tag{21}$$

The boundary of this semispace is simply the point $(-\infty, ..., -\infty)$. Since we obviously can not split a point we are left with only two trival hemispaces. Either this boundary point is with the semispace, and we have the whole space, or it is not with the semispace and we have $S(-\infty)$ and $\complement S(-\infty)$ as our two hemispaces.

Now consider the following hyperplane boundary:

$$x_0 \oplus ... \oplus x_i \oplus a_{i+1}x_{i+1} \oplus ... \oplus a_mx_m = x_{m+1} \oplus ... \oplus x_l \oplus a_{l+1}x_{l+1} \oplus ... \oplus a_jx_j \tag{22}$$

Where $i < m < l < j$ and $a_k = -\infty \ \forall k$

This is a hyperplane boundary that is missing some variables and has some coeffiecents equal to $-\infty$. First follow the rule for having coefficents equal to $-\infty$ and take the $n - ((m-i) + (j-l))$ dimensional cross section where $x_{i+1} = ... = x_m = x_{l+1} = ... = x_j = -\infty$. Then follow the rule for not having all the variables. So the cross section goes from acting like a $\mathbb{R}^{i+(l-m)+(n-j)}_{\max}$ space to a $\mathbb{R}^{i+(l-m)}_{\max}$ space.

8 References

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